A New Class of (2+1)-Dimensional Combined Structures with Completely Elastic and Non-Elastic Interactive Properties

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Z. Naturforsch. **61a,** 53 – 59 (2006); received September 24, 2005

Taking the new (2+1)-dimensional generalized Broer-Kaup system as an example, we obtain an exact variable separation excitation which can describe some quite universal (2+1)-dimensional physical models, with the help of the extended homogeneous balance method. Based on the derived excitation, a new class of combined structures, i. e., semifolded solitary waves and semifoldons, is defined and studied. The interactions of the semifolded localized structures are illustrated both analytically and graphically. – PACS numbers: 05.45.Yv, 02.30.Jr, 02.30.Ik

Key words: (2+1)-Dimensional; Semifolded Solitary Waves; Semifoldons; Interaction; Localized Excitations.

1. Introduction

In nonlinear science, soliton theory plays an essential role and has been applied in almost all natural sciences, especially in all physics branches such as condensed matter physics, field theory, fluid dynamics, plasma physics, optics [1]. Recently, it has been found that a quite "universal" formula [2 – 4],

$$u \equiv \frac{\lambda(a_1a_2 - a_0a_3)p_xq_y}{(a_0 + a_1p + a_2q + a_3pq)^2}, \tag{1}$$

is valid for suitable fields or potentials of various (2+1)-dimensional physically interesting integrable models including the Davey-Stewartson (DS) equation, the dispersive long wave equation (DLWE) [2], the Broer-Kaup (BK) system [2, 5], the higher-order Broer-Kaup (HBK) system [3], the Nizhnik-Novikov-Vesselov (NNV) system, the ANNV (asymmetric NNV) equation, and so on [2]. In (1), $p \equiv p(x,t)$ is an arbitrary function of $\{x,t\}$, $q \equiv q(y,t)$. It may be either an arbitrary function for some kinds of models such as the DS equation, or an arbitrary solution of a Riccati equation for some others, say the DLWE, while a_0, a_1, a_2, and a_3 are taken as constants. One of the most important results obtained from (1) is that for all models mentioned above there are quite rich localized excitations. Most of the previous studies on soliton theory, especially in higher dimensions, are restricted to single-valued situations, such as dromion (exponentially localized in all directions), compacton [this type of solutions describes the typical (1+1)-dimensional soliton solutions with finite wavelength when the nonlinear dispersion effects are included in the models], peakon [a special type of weak solutions of the (1+1)-dimensional Camassa-Holm (CH) equation being discontinuous at their crest] and their interactions. However, in various cases, the real natural phenomena are too intricate to be described only by single-valued functions. For instance, in nature there exist very complicated folded phenomena such as the folded protein [6], folded brain and skin surface, and many other kinds of folded biological systems [7]. The simplest multi-valued (folded) waves may be the bubbles on (or under) a fluid surface. Various ocean waves are really folded waves, too. In [2 – 4, 8], the authors discussed some simple cases of multiple valued solitary waves (folded in all directions). The properties of the interactions among different types of solitary waves like peakons, dromions, and compactons were discussed both analytically and graphically in [9]. However, nature is colorful and may exhibit quite complicated structures such as semifolded ones, when the function q is a single-valued function and p is selected via the relations

$$p_x = \sum_{i=1}^{M} U_i(\xi + w_it),$$

$$x = \xi + \sum_{i=1}^{M} X_i(\xi + w_i t),$$
$$p = \int^{\xi} p_x x_\xi \mathrm{d}\xi, \quad (2)$$

where U_i and X_i are localized excitations with the properties $U_i(\pm\infty) = 0, X_i(\pm\infty) = \text{const}$. From (2) one knows that ξ may be a multi-valued function in some suitable regions of x by selecting the functions X_i appropriately. Therefore, the function p_x, which is obviously an interaction solution of M localized excitations because of the property $\xi|_{x\to\infty} \to \infty$, may be a multi-valued function of x in these areas, though it is a single-valued functions of ξ. Actually, most of the known multi-loop solutions are a special situation of (2). For convenience later, we define the above localized excitations as semifolded solitary waves (SFSWs). Furthermore, if the interactions among the semifolded solitary waves are completely elastic, we call them semifoldons. To study the semifolded localized structures and their interactions in a (2+1)-dimensional system, we take the new (2+1)-dimensional generalized Broer-Kaup (GBK) system

$$\begin{aligned} & h_t - h_{xx} + 2hh_x + u_x + Au + Bg = 0, \\ & g_t + 2(gh)_x + g_{xx} + 4A(g_x - h_{xy}) \\ & \quad + 4B(g_y - h_{yy}) + C(g - 2h_y) = 0, \\ & u_y - g_x = 0, \end{aligned} \quad (3)$$

where A, B, and C are arbitrary constants, as a concrete example. In [10], Zhang et al. gave some special exact solutions of (3) by using the truncated Painlevé expansion. In this paper, by using the variable separation approach based on the extended homogeneous balance method [11], we obtain a general variable separation solution.

The paper is organized as follows. In Section 2, we apply a variable separated approach (VSA) based on the extended homogeneous balance method (EHBM) to solve the (2+1)-dimensional GBK and obtain its exact excitation. Section 3 is devoted to investigating the interaction properties both for the semifoldons and between single-valued and semifolded localized excitations. A brief discussion and summary is given in the last section.

2. Variable Separated Solutions of the (2+1)-Dimensional GBK Equation

According to the EHBM, let

$$h = f(\varphi)_x + h_0, \; g = v(\varphi)_{xy} + g_0, \; u = w(\varphi)_{xx} + u_0, \quad (4)$$

where $f(\varphi)$, $v(\varphi)$, and $w(\varphi)$ are functions of one argument $\varphi = \varphi(x,y,t)$ only, and $\{h_0, g_0, u_0\}$ are arbitrary known seed solutions of the GBK equation. For simplicity, we fix the seed solution as [10]

$$h_0 = h_0(x,t), \quad g_0 = 0, \quad (5)$$

$$u_0 = \mathrm{e}^{-Ax}\left[F_1(t) + \int^{x} \mathrm{e}^{Ax}(h_{0xx} - 2h_0 h_{0x} - h_{0t})\mathrm{d}x\right],$$

where $h_0(x,t)$ and $F_1(t)$ are arbitrary functions of $\{x,t\}$ and $\{t\}$, respectively.

Substituting (4) with (5) into (3) and canceling all the coefficients of the different powers in the partial derivatives of $\varphi(x,y,t)$, we have

$$v = w = 2f = 2\ln\varphi, \quad (6)$$

and the function φ should be a solution of

$$\varphi_t + 2(A + h_0)\varphi_x + 2B\varphi_y + 2\varphi_{xx} = 0. \quad (7)$$

To obtain some explicit solution of (7), we suppose that φ has the following variable separation form:

$$\varphi = p(x,t) + q(y,t). \quad (8)$$

Substituting (8) into (7), one finds that

$$q(y,t) = F_2(y - 2Bt) - \theta(t), \quad (9)$$

with $F_2(y - 2Bt) \equiv F_2$ and $\theta(t) \equiv \theta$ being arbitrary functions of the indicated variables, while $p(x,t)$ may be remained as arbitrary function of $\{x,t\}$ when h_0 is fixed as

$$h_0 = -\frac{1}{p_x}(p_t - \theta_t + p_{xx} + 2Ap_x). \quad (10)$$

Substituting all the results into (4), we obtain the corresponding exact solution of (3). Especially we are interested in the structure of a solution for the field g which has the final form

$$g = -2\frac{p_x q_y}{(p+q)^2}, \quad (11)$$

with p being an arbitrary function and q being given by (9). From (11) we can see that it possesses the same form of the "universal" formula (1) with $a_0 = a_3 = 0$, $a_1 = a_2 = 1$, $\lambda = -2$.

3. A New Interaction Property of Localized Structures for the (2+1)-Dimensional GBK System

In order to discuss the interaction property of localized excitations related to the physical quantity (11) [or (1)], we first study the asymptotic behavior of the localized excitations produced from (11) when $t \to \mp\infty$.

3.1. Asymptotic Behaviors of the Localized Excitations Produced from (11) [8, 9]

In general, if the functions p and q are selected as localized solitonic excitations with

$$p|_{t\to\mp\infty} = \sum_{i=1}^{M} p_i^{\mp}, p_i^{\mp} \equiv p_i(x - c_i t + \delta_i^{\mp}), \quad (12)$$

$$q|_{t\to\mp\infty} = \sum_{j=1}^{N} q_j^{\mp}, q_j^{\mp} \equiv q_j(y - C_j t + \Delta_j^{\mp}), \quad (13)$$

where $\{p_i, q_j\} \forall i$ and j are localized functions, then the physical quantity g, expressed by (11), delivers $M \cdot N$ (2+1)-dimensional localized excitations with the asymptotic behaviour

$$g|_{t\to\mp\infty} \to \sum_{i=1}^{M}\sum_{j=1}^{N} \left\{ \frac{-2p_{ix}^{\mp} q_{jy}^{\mp}}{\left((p_i^{\mp} + X_i^{\mp}) + (q_j^{\mp} + \Phi_j^{\mp})\right)^2} \right\}$$
$$\equiv \sum_{i=1}^{M}\sum_{j=1}^{N} g_{ij}^{\mp}(x - c_i t + \delta_i^{\mp}, y - C_j t + \Delta_j^{\mp})$$
$$\equiv \sum_{i=1}^{M}\sum_{j=1}^{N} g_{ij}^{\mp}, \quad (14)$$

where

$$X_i^{\mp} = \sum_{j<i} p_j(\mp\infty) + \sum_{j>i} p_j(\pm\infty), \quad (15)$$

$$\Phi_i^{\mp} = \sum_{j<i} q_j(\mp\infty) + \sum_{j>i} q_j(\pm\infty), \quad (16)$$

and we have assumed without loss of generality, $C_i > C_j$ and $c_i > c_j$ if $i > j$.

It can be deduced from (14) that the ijth localized excitation g_{ij} preserves its shape during the interaction if

$$X_i^{+} = X_i^{-}, \quad (17)$$

$$\Phi_j^{+} = \Phi_j^{-}. \quad (18)$$

Meanwhile, the phase shift of the ijth localized excitation g_{ij} reads

$$\delta_i^{+} - \delta_i^{-} \quad (19)$$

in the x direction and

$$\Delta_j^{+} - \Delta_j^{-} \quad (20)$$

in the y direction.

The above discussions demonstrate that localized solitonic excitations for the universal quantity g [(11) or (1)] can be constructed without difficulties via the (1+1)-dimensional localized excitations with the properties (12), (13), (17), and (18). As a matter of fact, any localized solutions (or their derivatives) with completely elastic (or not completely elastic or completely inelastic) interaction behaviors of any known (1+1)-dimensional integrable model can be utilized to construct (2+1)-dimensional localized solitonic solutions with completely elastic ($X_i^{+} = X_i^{-}$, $\Phi_j^{+} = \Phi_j^{-}$ for all i, j) or not completely elastic or completely inelastic ($X_i^{+} \neq X_i^{-}$, $\Phi_j^{+} \neq \Phi_j^{-}$ at least for one of i, j) interaction properties. However, to the best of our knowledge the interactions among semifoldons, peakons, dromions, and compactons were little reported in the literature. In order to see the interaction behaviors among them more directly and visually, we investigate some special examples by fixing the arbitrary functions p and q in (11).

3.2. Completely Elastic Interactions

Now we discuss some new coherent structures for the physical quantity g, and focus our attention on some (2+1)-dimensional semifolded localized structures, which may exist in certain situations, when the function q is a single-valued function and p is selected as (2) with $N = 2$, namely

$$p_x = \frac{4}{5}\text{sech}^2(\xi) + \frac{1}{2}\text{sech}^2(\xi - 0.3t),$$
$$x = \xi - 1.5\tanh(\xi) - 1.5\tanh(\xi - 0.3t), \quad (21)$$

$$q = 1 + \exp(y - 2Bt). \quad (22)$$

Then we successfully construct semifolded localized excitations that possess phase shifts for the physical quantity g depicted in Figure 1. From Fig. 1 we can see that the two semifolded localized excitations possess novel properties, which fold in the x direction and

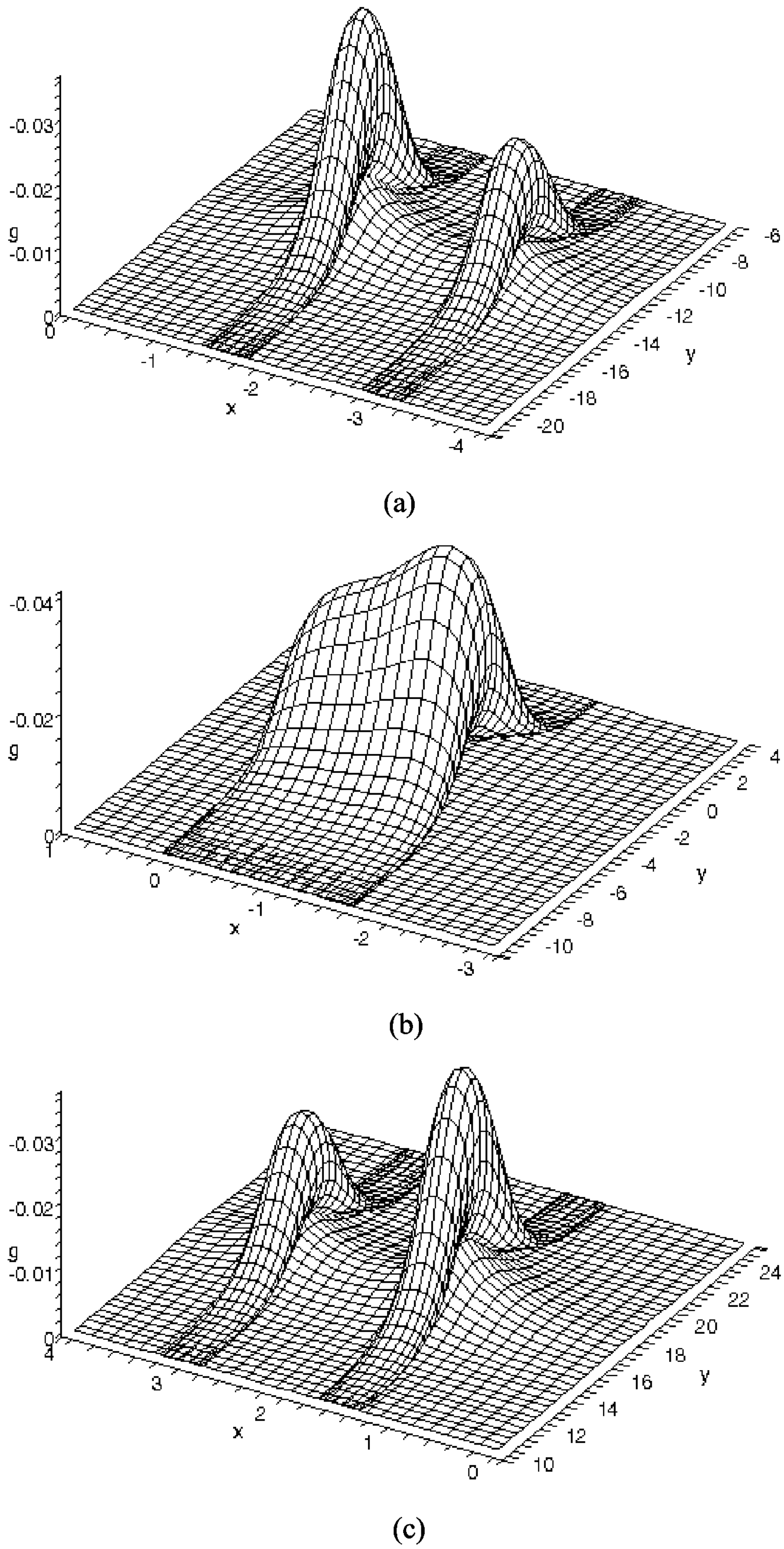

Fig. 1. The evolution of the interactions of two semifolded localized structures for the physical quantity g expressed by (11) with the conditions (21) and (22) at the times (a) $t = -15$, (b) $t = -5$, (c) $t = 15$.

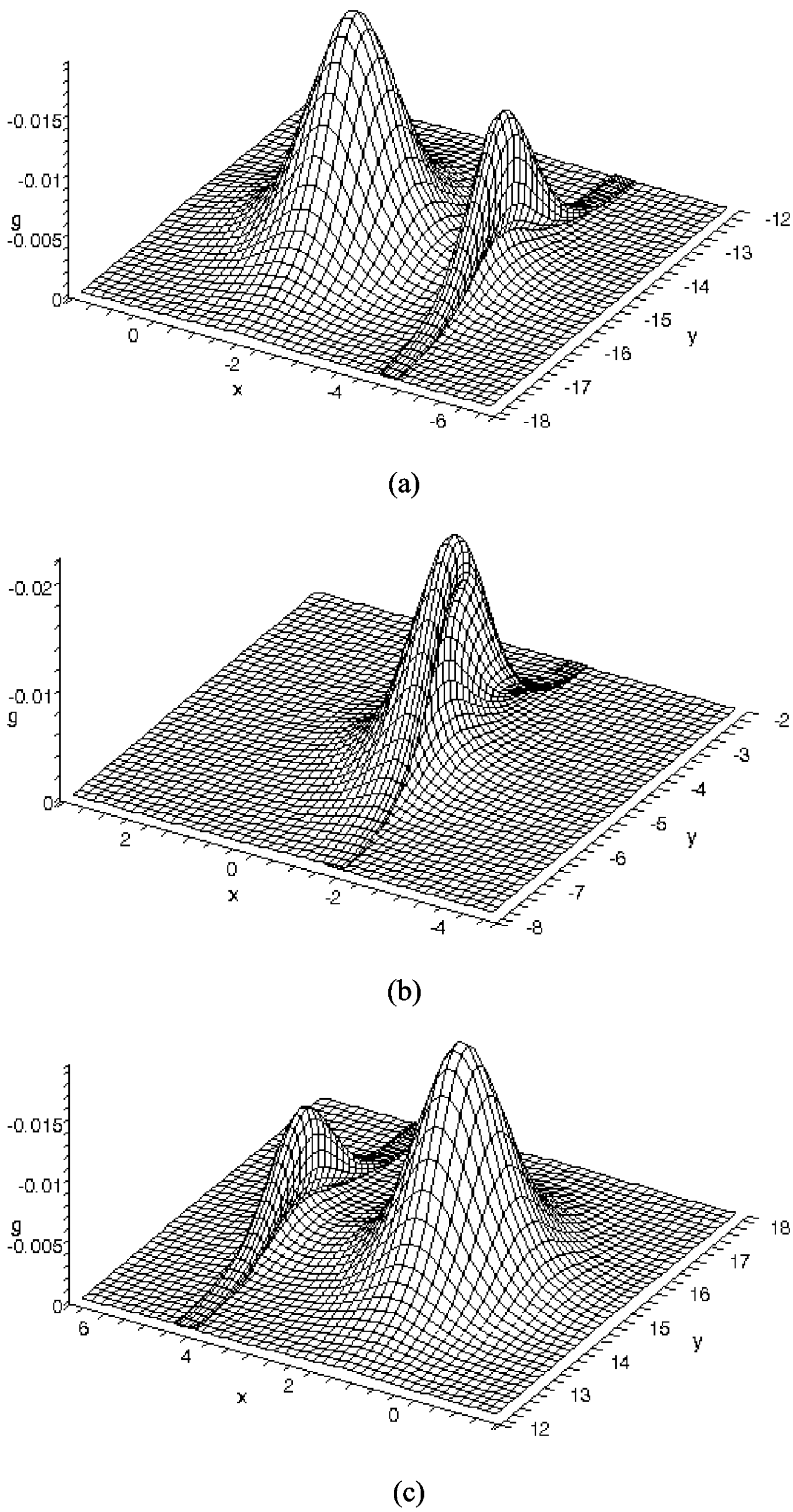

Fig. 2 The evolution of the interactions between semifoldon and dromion for the physical quantity g expressed by (11) with the conditions (26) and (27) at the times (a) $t = -15$, (b) $t = -5$, (c) $t = 15$.

localize in a usual single valued way in the y direction. Moreover, one can find that the interaction between the two semifolded localized excitations (semifoldons) is completely elastic, which is very similar to the completely elastic collisions between two classical particles, since the velocity of one of the localized structures has to be set to zero and there are still phase shifts for the two semifolded localized excitations. To see this more carefully, one can easily find that the position located by the large static localized structure is altered from about $x = -1.5$ to $x = 1.5$, and its shape is completely preserved after the interaction.

Along the same line of argument and performing a similar analysis, when p and q are taken of the forms

$$\begin{aligned} p_x &= \frac{4}{5}\operatorname{sech}^2(\xi) + \frac{1}{2}\operatorname{sech}^2(\xi - 0.3t), \\ x &= \xi - 1.5\tanh(\xi) - 1.5\tanh(\xi - 0.3t), \end{aligned} \tag{23}$$

$$q = \begin{cases} 0, & \text{if } y - 2Bt \le -\dfrac{\pi}{2}, \\ \sin(y - 2Bt) + 1, & \text{if } -\dfrac{\pi}{2} < y - 2Bt \le \dfrac{\pi}{2}, \\ 2, & \text{if } y - 2Bt > \dfrac{\pi}{2}, \end{cases} \tag{24}$$

or

$$q = \begin{cases} \exp(y - 2Bt), & \text{if } y - 2Bt \le 0, \\ -\exp(-y + 2Bt), & \text{if } y - 2Bt > 0, \end{cases} \tag{25}$$

we may construct other two types of semifolded localized structures for the physical quantity g, and find that their interaction is also completely elastic.

3.3. Not-Completely Elastic Interactions

It is interesting to mention that, though the above choices lead to completely elastic interactions for the (2+1)-dimensional solutions, one can also derive some combined localized coherent structures with not-completely elastic interactions by selecting p and q appropriately. One of the simple choices of combined localized coherent structures with not-completely elastic interaction is

$$\begin{aligned} p_x &= \frac{4}{5}\operatorname{sech}^2(\xi) + \frac{1}{2}\operatorname{sech}^2(\xi - 0.3t), \\ x &= \xi - 1.5\tanh(\xi - 0.3t), \end{aligned} \tag{26}$$

$$q = \tanh(y - 2Bt). \tag{27}$$

The corresponding interaction behaviors are depicted in Figure 2. From Fig. 2, we can find that the interaction between semifoldons and dromions may exhibit a novel property, which is not-completely elastic since the shapes are not completely preserved after the interaction.

In fact, we can also construct combined semifoldon-compacton and semifoldon-peakon localized coherent structures with not-completely elastic interaction by selecting p and q as

$$\begin{aligned} p_x &= \frac{4}{5}\operatorname{sech}^2(\xi) + \frac{1}{2}\operatorname{sech}^2(\xi - 0.3t), \\ x &= \xi - 1.5\tanh(\xi - 0.3t), \end{aligned} \tag{28}$$

$$q = \begin{cases} 0, & \text{if } y - 2Bt \le -\dfrac{\pi}{2}, \\ \sin(y - 2Bt) + 1, & \text{if } -\dfrac{\pi}{2} < y - 2Bt \le \dfrac{\pi}{2}, \\ 2 & \text{if } y - 2Bt > \dfrac{\pi}{2}, \end{cases} \tag{29}$$

or

$$q = \begin{cases} \exp(y - 2Bt), & \text{if } y - 2Bt \le 0, \\ -\exp(-y + 2Bt), & \text{if } y - 2Bt > 0. \end{cases} \tag{30}$$

The corresponding evolution plot is omitted here.

4. Summary

Starting from the obtained variable separated excitations which describe a quite universal (2+1)-dimensional physical model of a (2+1)-dimensional GBK system, we deal with the interactions among semifoldons, peakons, dromions, and compactons both analytically and graphically, and reveal some novel and interesting properties: The interactions among semifoldons that possess phase shifts are completely elastic, and the interactions of semifoldon-dromion, semifoldon-compacton, and semifoldon-peakon are not-completely elastic, depending on the specific details of the solutions. Because of the complexity of folded phenomena and the wide applications of the soliton theory, to learn more about the new localized structures and interactions between different types of solitary waves and their applications is worth further study.

Acknowledgement

The project is supported by the National Natural Science Foundation of China and the Natural Science Foundation of Shandong Province in China.

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